Theorem

Let $f(x)$ be a cubic polynomial such that $f'(x)$ is never 0, then any initial guess will force Newton's method to converge to the zero of f.

Proof

We start with some simplifications. First note that if Newton's Method does not converge for the function $f(x)$ with initial guess x_1, then it will also not converge for the function $f(x-r)$ with initial guess $x_1 - r$. That is, shifting the function and the initial guess to the left or right will not change whether Newton's Method converges of diverges. Therefore, it is enough to prove this theorem for functions that pass through the origin. Next, if $f'(x)$ is never zero, then $f'(x)$ is either always positive or always negative. If $f'(x)$ is always negative and Newton's Method diverges with initial guess x_1 then Newton's method will also diverge for the positive function $-f(x)$ with initial guess x_1. This is true because the divergence of Newton's Method does not change under reflections across the x-axis. Thus, it is enough to prove the theorem for $f'(x) > 0$. Let

$$f(x) = ax^3 + bx^2 + cx$$

Then

$$f'(x) = 3ax^2 + 2bx + c$$

and

$$f''(x) = 6ax + 2b$$

By Larson page 232, we need to show that

$$\left| \frac{f(x) f''(x)}{\left(f'(x) \right)^2} \right| < 1$$

or that

$$\left| f(x) f''(x) \right| < \left(f'(x) \right)^2$$

Case 1: $f''(x) > 0$ which is equivalent to $3ax > -b$

We need to show

$$f(x) f''(x) < \left(f'(x) \right)^2$$

or that

$$\left(ax^3+bx^2+cx\right)\left(6ax+2b\right)<\left(3ax^2+2bx+c\right)^2$$

After multiplying out and bringing all the terms to the right hand side and switching, we get that this is equivalent to

$$3a^2x^4+4abx^3+2b^2x^2+2bcx+c^2>0$$

or

$$3a^2x^4+3abx^3+abx^3+2b^2x^2+2bcx+c^2>0$$

or

$$\left(3ax\right)\left(ax^3+bx^2\right)+abx^3+2b^2x^2+2bcx+c^2>0$$

Since $3ax>-b$ we need to show

$$(-b)\left(ax^3+bx^2\right)+abx^3+2b^2x^2+2bcx+c^2\geq 0$$

This simplifies to

$$b^2x^2+2bcx+c^2\geq 0$$

Factor to get

$$(bx+c)^2\geq 0$$

which is true since a square cannot be negative.

Case 2: $f''(x)<0$ which is equivalent to $-3ax<b$

We need to show

$$-f(x)f''(x)<\left(f'(x)\right)^2$$

or that

$$-\left(ax^3+bx^2+cx\right)\left(6ax+2b\right)<\left(3ax^2+2bx+c\right)^2$$

After multiplying out and bringing all the terms to the right hand side and switching, we get that this is equivalent to

$$15a^2x^4 + 20abx^3 + \left(6b^2 + 12ac\right)x^2 + 6bcx + c^2 > 0$$

Since $-3ax > b$ we get

$$15a^2x^4 + 20a\left(-3ax\right)x^3 + \left(6\left(-3ax\right)^2 + 12ac\right)x^2 + 6\left(-3ax\right)cx + c^2 \geq 0$$

This simplifies to

$$9a^2x^4 - 6acx + c^2 \geq 0$$

This factors as

$$\left(3ax^2 - c\right)^2 \geq 0$$

The left hand side is a square which cannot be negative. Hence, the inequality holds.

Notice that if the second derivative equals 0, then the left hand side of Larson's inequality and 0, so the fraction is 0 which is less than 1. Therefore, no matter what the second derivative is, Larson's inequality holds. Thus Newton's Method always converges.